Homework2

Problem 1. Partial integral

1. Prove

$$\iiint f(\nabla\cdot\vec{A})\mathrm{d}v = \oint f\vec{A}\cdot\mathrm{d}\vec{a} - \iiint(\vec{A}\cdot\nabla f)\mathrm{d}v$$

   where $\iiint dv$ means a volume integral, and $\oint \mathrm{d}\vec{a}$ is the integral over the surface surrounding this volume. f is a scalar function, and $\vec{A}$ is a vector function.
2. We extend the volume integral to the entire space. In many physical applications, $f\vec{A}$ decays faster than $1/r^2$ as $r\to\infty$. Simplify the above result in this case.

Problem 2. Spherical and cylindrical coordinates

1. The unit vectors of the spherical coordinates are denoted as $(\hat{e}_r, \hat{e}_\theta, \hat{e}_\phi)$. Please work out the relation between

$$(\hat{e}_r, \hat{e}_\theta, \hat{e}_\phi) = (\hat{e}_x, \hat{e}_y, \hat{e}_z)O$$

   where O is a 3×3 orthogonal matrix. Write the explicit form of O.
2. Work out the relation

$$(\mathrm{d}\hat{e}_r, \mathrm{d}\hat{e}_\theta, \mathrm{d}\hat{e}_\phi) = (\hat{e}_r, \hat{e}_\theta, \hat{e}_\phi)T$$

   where T is a 3×3 antisymmetric matrix. Write the explicit form of T .
3. Now we change to the cylindrical coordinates. Work out the relation between the unit vectors of the cylindrical coordinates and those of the Cartesian coordinates.

$$(\hat{e}_\rho, \hat{e}_\varphi, \hat{e}_z) = (\hat{e}_x, \hat{e}_y, \hat{e}_z)O'$$

   Work out the relation

$$(\mathrm{d}\hat{e}_\rho, \mathrm{d}\hat{e}_\psi, \mathrm{d}\hat{e}_z) = (\hat{e}_\rho, \hat{e}_\phi, \hat{e}_z)T'$$

   Write down the explicit forms of O' and T'.

Problem 3. Vector calculus in the curvilinear coordinates

Under the spherical coordinate, $\mathrm{d}\vec{r} = \mathrm{d}r\hat{e}_r + r\mathrm{d}\theta\hat{e}_\theta + r\sin\theta\mathrm{d}\phi\hat{e}_\phi$. Under the cylindrical coordinate, $\mathrm{d}\vec{r} = \mathrm{d}\rho\hat{e}_\rho + \rho\mathrm{d}\psi\hat{e}_\psi + \mathrm{d}z\hat{e}_z$. The formula of the gradient for a scalar function $f(u^1, u^2, u^3)$ in orthogonal curvilinear coordinates is

$$\nabla f = \sum_i \frac{1}{\sqrt{g_{ii}}}\frac{\partial}{\partial u^i} f\ \vec{e}_{u^i}$$

For a vector field $\vec{v} = v^1\hat{e}_{u^1} + v^2\hat{e}_{u^2} + v^3\hat{e}_{u^3}$, its divergence is

$$\nabla\cdot\vec{v} = \frac{1}{\sqrt{|detg|}}\sum_i \frac{\partial}{\partial u^i}\sqrt{|\frac{detg}{g_{ii}}}|v^i$$

(The Einstein notation is not assumed here.)

1. Write down the metric matrix g for the spherical and cylindrical coordinates, such as $(\mathrm{d}\vec{s})^2 = g_{ij}\mathrm{d}u^i\mathrm{d}u^j$ (Einstein notation is assumed, and $\mathrm{d}\vec{s}$ is the distense of tow vector in different coordinates which will not change.). For example in spherical coordinates: $\mathrm{d}u^1 = \mathrm{d}r, \mathrm{d}u^2 = \mathrm{d}\theta, \mathrm{d}u^3 = \mathrm{d}\phi$.
2. Write down the volume element dv in terms of the spherical and cylin- drical coordinates.
3. Work out the expression of the Laplacian ∇^2 operator in terms of the spherical coordinates and the cylindrical coordinates.